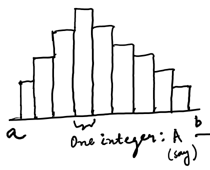
Lecture 18

Monday, July 10, 2017 12:59 PM

Chapter 4: (Continuous RV)

X = The depth of the lake at a random location.

Measure X at n locations (in meters).



$\rightarrow$ Height of that bar $\propto \dfrac{n_A}{n}$

$n_A \rightarrow$ # locations where the depth is A meters

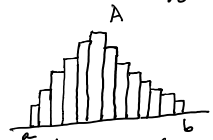
$n \rightarrow$ Total # of locations

Relative freq. $\rightarrow$ Prob.

Area of that bar $= \dfrac{n_A}{n} \times 1$ sq.units

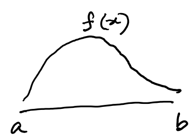
Total area of all bars $= \sum_A \dfrac{n_A}{n} = 1$ (sq. units)

We increase precision:
(precision $\uparrow \infty$, $n \uparrow \infty$)



(Total area is still 1)

Top of the histogram approaches a smooth curve, which is called the prob. density curve and the corresponding function is called the probability density function (pdf).
(f)



Here, Total area under the curve (between a & b meters)
$= 1$

So, $\int_a^b f(x)\,dx = 1$

Here, Support of X (or f): $[a, b]$

In general, $D_X = \{x : f_X(x) > 0\}$

So, $f_X(x) = 0$ outside D_X (We can't observe values from there)

Evidently, $f_X(x) \geq 0 \ \forall x$.

So, we can say: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ or $\int_{\mathbb{R}} f_X(x)\,dx = 1$ (This is equivalent to $\int_{D_X} f_X(x)\,dx = 1$)

Requirements:

1. $f(x) \geq 0 \ \forall x \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{D_X} f_X(x)\,dx = 1$

Discrete	vs. Conts.
1. D is finite or countably infinite eg. $D = \{1,2,3\}$ or $D = \{1,2,3,\dots\}$	1. D is uncountably infinite. eg. $D = [0,1]$ or $[0,1] \cup [2,3]$
2. PMF: $p(x) \geq 0 \ \forall x$ s.t. $\sum_{x \in D} p(x) = 1$	2. PDF: $f(x) \geq 0 \ \forall x$ s.t. $\int_{x \in D} f(x)\,dx = 1$

b

2. PMF: $p(x) \geq 0 \ \forall x$ s.t. $\sum_{x \in D} p(x) = 1$ | s.t. $\int_{x \in D} f(x)\,dx = 1$

We have also seen: $P(a \leq X \leq b) = 1 = \int_a^b f(x)\,dx$

In general, for any a & b s.t. $a \leq b$,

$$P(a \leq X \leq b)$$
$$= \text{Area under the curve } f \text{ between } a \text{ and } b$$
$$= \int_a^b f(x)\,dx$$

$$P(a \leq X \leq b) = \frac{n_a + n_{a+1} + \dots + n_b}{n}$$
$$= \frac{n_a}{n} + \frac{n_{a+1}}{n} + \dots + \frac{n_b}{n}$$

Clearly, $0 \leq P(a \leq X \leq b) \leq 1$ $\forall a, b$ s.t. $a \leq b$.

$$P(X = c) = \int_{\{x=c\}} f(x)\,dx = \int_c^c f(x)\,dx = \lim_{\varepsilon \to 0} \int_{c-\varepsilon}^{c+\varepsilon} f(x)\,dx = 0$$

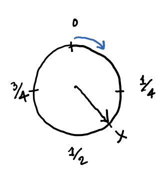
As a result:

1. $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$
$$= \int_a^b f(x)\,dx$$

[Pf: The first 2 terms: $P(a \leq X \leq b) = P(X = a) + P(a < X \leq b)$
$= 0 + P(a < X \leq b)$]

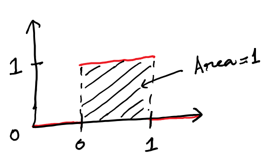
2. $P(X \leq a) = P(X < a) = 1 - P(X \geq a) = 1 - P(X > a) = \int_{-\infty}^{a} f(x)\,dx = 1 - \int_a^{\infty} f(x)\,dx.$

Eg. 1.



X = a "random" number in $[0,1)$
f: density fn. of X.
For an unbiased spin, $f(x) = c$ on $[0,1)$ for some c. $(c > 0)$

$$\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 c \cdot dx = 1 \Rightarrow c \cdot [x]_0^1 = 1$$
$$\Rightarrow c(1-0) = 1$$
$$\Rightarrow c = 1$$



X has a Uniform dist^n on $[0,1)$.

In general, if measurement is between a and b.

1. $f(x) = c_1$ on $[a,b)$ $(c_1 > 0)$

2. $\int_a^b f(x)\,dx = \int_a^b c_1 \cdot dx = c_1(b-a) = 1$
$$\Rightarrow c_1 = \frac{1}{b-a}.$$

This is the uniform dist^n on $[a,b)$.

Lecture 19

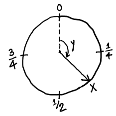
Tuesday, July 11, 2017 1:00 PM

- $X \sim \mathcal{U}(\underbrace{a,b}_{\text{parameters}})$ if

$$f(x) = (b-a)^{-1} I_{[a,b)}(x)$$

Indicator fn: $I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

Eg. 1 (contd)



Y = angle that the spinning arm makes with "0". (in degrees)

So, $Y \sim \mathcal{U}(0,360)$ [Using same logic as X]

$$P(90 \le Y \le 180) = \int_{90}^{180} \frac{1}{360} \cdot dy = \frac{180-90}{360} = \frac{1}{4}$$

$$P(0 \le Y \le 90 \text{ or } 180 \le Y \le 270) = \frac{90-0}{360} + \frac{270-180}{360} = \frac{1}{2}$$

Clearly, $Y = 360X$.

2. $\underbrace{f(x; K, \theta)}_{\text{Pareto density}} \propto \frac{1}{x^{K+1}} I_{[\theta,\infty)}(x)$

Pareto density (2-parameter family)

$K \geq 1$ integer

$\theta > 0$

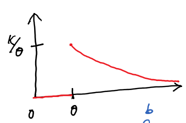
It's easy to see: $f(x) \geq 0 \ \forall x$

Let c be the proportionality const.

$$\int_{\mathbb{R}} f(x)\,dx = \int_D f(x)\,dx = \int_\theta^\infty \frac{c\,dx}{x^{K+1}} = c \cdot \left[\frac{x^{-K}}{-K}\right]_\theta^\infty = \frac{c}{K}\left(\frac{1}{x^K}\right)_\infty^\theta$$

$$= \frac{c}{K\theta^K} = 1$$

$\Rightarrow c = K\theta^K$ (This makes f a density)



$$f(x) = \begin{cases} 0, & \text{if } x < \theta \\ \frac{K\theta^K}{x^{K+1}}, & \text{if } x \geq \theta \end{cases} = \frac{K\theta^K}{x^{K+1}} I_{[\theta,\infty)}(x)$$

$$P(X \le b) = \begin{cases} \int_{-\infty}^b 0 \cdot dx = 0, & \text{if } b \le \theta \\ \int_{-\infty}^b f(x)\,dx = \int_\theta^b \frac{K\theta^K}{x^{K+1}}\,dx, & \text{if } b > \theta \end{cases}$$

$$= K\theta^K \left[\frac{1}{Kx^K}\right]_b^\theta$$

$$= \theta^K \cdot \left(\frac{1}{\theta^K} - \frac{1}{b^K}\right)$$

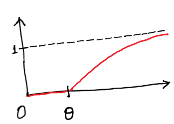
$$= 1 - \left(\frac{\theta}{b}\right)^K$$

Try to calculate $P(a \le X \le b)$ for different values $a \le b$ (compared to θ)

- Cumulative Dist^n Function:

$$\text{CDF} := F(x) = \int_{-\infty}^x f(y)\,dy$$

e.g. 1. For Pareto (K,θ), $F(x) = \begin{cases} 0, & \text{if } x \le \theta \\ 1 - \left(\frac{\theta}{x}\right)^K, & \text{if } x > \theta \end{cases}$



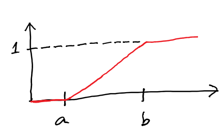
Properties of CDF

- $F(x) = \int_{-\infty}^x f(y)\,dy \Rightarrow F'(x) = f(x)$
- $F(x) \downarrow 0$ as $x \downarrow -\infty$
- $F(x) \uparrow 1$ as $x \uparrow \infty$
- F is non-decreasing
- F is conts.
- F uniquely identifies a dist^n (just like f)

$$\lim_{x\to\infty}\left(\frac{\theta^K}{x^K}\right) = \theta^K \cdot \lim_{x\to\infty} \frac{1}{x^K} = 0$$

$$\Rightarrow \lim_{x\to\infty} F(x) = 1$$

eg. 2. For $\mathcal{U}(a,b)$, $F(x) = \begin{cases} 0, & \text{if } x < a \\ \int_{-\infty}^x f(z)\,dz = \int_a^x \frac{1}{b-a} \cdot dx = \frac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x > b \end{cases}$



- If $X \sim \mathcal{U}(0,1)$, then $Y = a + (b-a) \cdot X \sim \mathcal{U}(a,b)$.

Pf: We know $D_X = (0,1)$ and CDF of X: $F_X(x) = \begin{cases} 0 & ; \text{if } x < 0 \\ x & ; \text{if } 0 \le x \le 1 \\ 1 & ; \text{if } x > 1 \end{cases}$ (Just put $a=0, b=1$ in eg. 2 above)

Clearly, $D_Y = (a,b)$ and CDF of Y: (for any $y \in D_Y$)

$$F_Y(y) = P(Y \le y) = P(a + (b-a) \cdot X \le y) = P\left(X \le \frac{y-a}{b-a}\right) = F_X\left(\frac{y-a}{b-a}\right) = \begin{cases} 0, & \text{if } \frac{y-a}{b-a} < 0, \text{ i.e. } y < a \\ \ldots & \ldots \frac{y-a}{\ldots} \le 1, \text{ i.e. } a \le y \le b \end{cases}$$

Pf: We know $D_X =$
Clearly, $D_Y = (a,b)$ and CDF of Y: (for any $y \in D_Y$)

$$F_Y(y) = P(Y \le y) = P(a + (b-a)\cdot X \le y) = P\left(X \le \frac{y-a}{b-a}\right) = F_X\left(\frac{y-a}{b-a}\right) = \begin{cases} 0, & \text{if } \frac{y-a}{b-a} < 0, \text{ i.e. } y < a \\ \frac{y-a}{b-a}, & \text{if } 0 \le \frac{y-a}{b-a} \le 1, \text{ i.e. } a \le y \le b \\ 1, & \text{if } \frac{y-a}{b-a} > 1, \text{ i.e. } y > b \end{cases}$$

Comparing with the CDF of $\mathcal{U}(a,b)$,
$Y \sim \mathcal{U}(a,b)$ [By last property of CDF above]
Because, $Y = 360 \cdot X$ in eg. 1, $Y \sim \mathcal{U}(0,360)$. [$a = 0$, $b = 360$]

- Percentiles:

$\eta(p) := (100p)^{th}$ percentile $[0 \le p \le 1]$

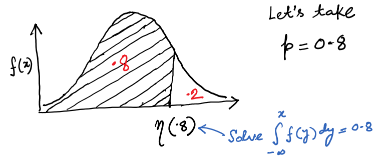
$\to$ $100(1-p)$% values are above $\eta(p)$.
$\to$ $100 \cdot p$% values are below $\eta(p)$.

$$\boxed{P(X \le \eta(p)) = p}$$

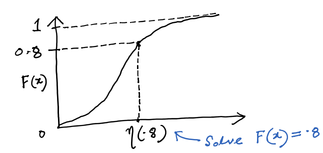
Defn of $(100p)^{th}$ percentile / $(4p)^{th}$ Quartile / p^{th} Quantile

Equivalently, $\int_{-\infty}^{\eta(p)} f(x)\,dx = p$ OR $F(\eta(p)) = p$

eg. $p = 0.5 \to \tilde{\mu}$: Median (Q_2)
$p = 0.25 \to Q_1$: 1^{st} Quartile
$p = 0.75 \to Q_3$: 3^{rd} Quartile



Let's take $p = 0.8$



We have seen: $F'(x) = f(x)$.

e.g. 1. $X \sim \mathcal{U}(a,b)$

$$F(x) = \begin{cases} 0, & \text{if } x < a \\ \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x > b \end{cases}$$

$$F'(x) = \begin{cases} 0, & \text{if } x<a \text{ or } x>b \\ \dfrac{1}{b-a}, & \text{if } a \le x \le b \end{cases} = (b-a)^{-1} I_{[a,b]}(x) = f(x)$$

2. $X \sim \text{Pareto}(\kappa, \theta)$ $[\kappa, \theta > 0]$

$$F(x) = \begin{cases} 0, & \text{if } x \le \theta \\ 1-\left(\dfrac{\theta}{x}\right)^{\kappa}, & \text{if } x > \theta \end{cases}$$

$$\left[\frac{d}{dx}\left[\left(\frac{\theta}{x}\right)^{\kappa}\right] = \theta^{\kappa}\frac{d}{dx}(x^{-\kappa}) = \theta^{\kappa}(-\kappa x^{-\kappa-1}) = -\frac{\kappa\theta^{\kappa}}{x^{\kappa+1}}\right]$$

$$F'(x) = \begin{cases} 0, & \text{if } x \le \theta \\ \dfrac{\kappa\theta^{\kappa}}{x^{\kappa+1}}, & \text{if } x>\theta \end{cases} = f(x)$$

• Percentiles: $F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(x)\,dx = p$

e.g. 1. $p = 0.5$, $\eta(.5) = \tilde{\mu}$.

$$F(x) = \begin{cases} 0, & \text{if } x \le \theta \\ 1-\left(\dfrac{\theta}{x}\right)^{\kappa}, & \text{if } x>\theta \end{cases}$$

Set $F(x) = 0.5 \Rightarrow 1-\left(\dfrac{\theta}{x}\right)^{\kappa} = 0.5$

$\Rightarrow \left(\dfrac{\theta}{x}\right)^{\kappa} = 0.5$

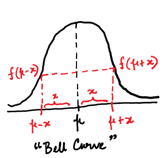
$\Rightarrow \dfrac{\theta}{x} = (0.5)^{1/\kappa}$

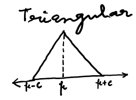
$\Rightarrow x = \dfrac{\theta}{\left(\frac{1}{2}\right)^{1/\kappa}} = (2)^{\frac{1}{\kappa}}.\theta = \sqrt[\kappa]{2}.\theta$

$$\boxed{\tilde{\mu} = \eta(.5) = \sqrt[\kappa]{2}.\theta}$$

• In general, if $f(x)$ is symmetric (about μ), then $\tilde{\mu} = \mu$.

Symmetry about μ:
$f(\mu+x) = f(\mu-x)$ $\forall x \in \mathbb{R}$
$\Rightarrow \tilde{\mu} = \mu$





Proof: $\int_{-\infty}^{\mu} f(x)\,dx = \int_{\infty}^{0} f(\mu-y)(-dy)$ [Putting $y = \mu - x$]

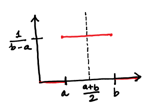
$= \int_{0}^{\infty} f(\mu-y)\,dy$

$= \int_{0}^{\infty} f(\mu+y)\,dy$ [By symmetry]

$= \int_{\mu}^{\infty} f(z)\,dz$ [Putting $z = \mu+y$]

Hence $\int_{-\infty}^{\mu} f(x)\,dx = \int_{\mu}^{\infty} f(x)\,dx$, but they add up to 1 $\Rightarrow \int_{-\infty}^{\mu} f(x)\,dx = 0.5 \Rightarrow \tilde{\mu} = \mu$ (Proved)

eg. 1. $X \sim \mathcal{U}(a,b)$
Symmetric about $\mu = \dfrac{a+b}{2}$



$F(\tilde{\mu}) = 0.5$

$\Rightarrow \dfrac{\tilde{\mu}-a}{b-a} = 0.5$

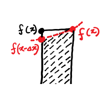
$\Rightarrow \tilde{\mu} = \dfrac{a+b}{2} = \mu$ (Check)

• Expectation / Mean:

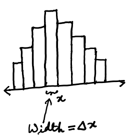
$$\int_{x-\Delta x}^{x} f(y)\,dy \approx f(x)\Delta x$$

For the discrete versions

$$\mu = \sum_{x\in D} x\,p(x) = \sum_{x \in D} x\,\underbrace{[F(x) - F(x-\Delta x)]}$$



Area of the shaded region ≈ Area of the rectangle
i.e. $\int^{x} f(y)\,dy \approx f(x).\Delta x$

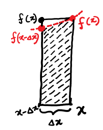


For the discrete version,

$$\mu = \sum_{x\in D} x\,p(x) = \sum_{x\in D} x\,[\overbrace{F(x)-F(x-\Delta x)}^{x-\Delta x}]$$

In the limit,

$$\mu = \lim_{\Delta x\to 0} \sum_{x\in D} x.f(x)\Delta x = \int_{x\in D} x f(x)\,dx$$



Area of the shaded region ≈ Area of the rectangle
i.e. $\int_{x-\Delta x}^{x} f(y)\,dy \approx f(x).\Delta x$

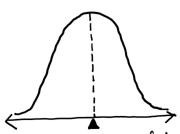
$$\boxed{\mu = \int_{x\in D} x f(x)\,dx}$$

eg. 1. $X \sim U(a,b)$

$$\mu \overset{Not^n}{:=} E(X) = \int_{x\in D} x f(x)\,dx = \int_a^b \frac{x}{b-a}\cdot dx = \frac{1}{b-a}\cdot\left[\frac{x^2}{2}\right]_a^b$$
$$= \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

Now, in general, if f is symmetric about μ, $E(X)=\mu$

Proof:
$$\int_{-\infty}^{\infty}(x-\mu)f(x)\,dx = \int_{-\infty}^{\mu}(x-\mu)f(x)\,dx + \int_{\mu}^{\infty}(x-\mu)f(x)\,dx$$
$$= \int_{-\infty}^{0} z f(\mu+z)\,dz + \int_{0}^{\infty} z f(\mu+z)\,dz \quad \left(\text{Putting } z=x-\mu\right)$$
$$= \int_{-\infty}^{0} z f(\mu+z)\,dz + \int_{0}^{-\infty}(-u)f(\mu-u)(-du) \quad \left(\text{Putting } u=-z \text{ in second term}\right)$$
$$= \int_{-\infty}^{0} z f(\mu+z)\,dz - \int_{-\infty}^{0} u f(\mu+u)\,du \quad (\text{By symmetry})$$
$$= 0$$
$$\Rightarrow \int_{-\infty}^{\infty} x f(x)\,dx = \mu\int_{-\infty}^{\infty} f(x)\,dx = \mu \quad (\text{Proved})$$



$E(X)$ is where the fulcrum should be placed to balance the horizontal axis if the entire density curve is placed on it. Clearly, if f is symmetric about μ, the fulcrum should be placed at μ

If not symmetric,

Right skewed / Positively skewed

$\mu > \tilde{\mu}$

Left skewed / Negatively skewed

$\mu < \tilde{\mu}$

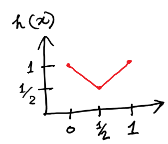
As the median $\tilde{\mu}$ is more 'robust' than the mean μ, μ is pulled more by the long tail (outliers) towards itself compared to $\tilde{\mu}$.

Defn: $E[h(X)] \overset{Not^n}{=} \mu_{h(X)} := \int_D h(x)f(x)\,dx = \int_{-\infty}^{\infty} h(x)f(x)\,dx$

$E[aX+b] = a\,E(X)+b$ (Check!)

e.g. $X \sim U(0,1)$

$$h(x) = \max\{x, 1-x\} = \begin{cases} 1-x, & \text{if } x \le 1-x \Leftrightarrow 2x\le 1 \Leftrightarrow x \le 1/2 \\ x, & \text{if } x > \frac{1}{2}\end{cases}$$



$$E[h(X)] = \int_0^1 h(x)\cdot f(x)\,dx = \int_0^1 h(x)\,dx = \int_0^{1/2} h(x)\,dx + \int_{1/2}^{1} h(x)\,dx$$
$$= \int_0^{1/2}(1-x)\,dx + \int_{1/2}^{1} x\,dx$$
$$= 3/4 \quad (\text{Check!})$$

- Variance: $\sigma_X^2 \overset{Not^n}{:=} V(X) := E\left[(X-\mu_X)^2\right]$

$$= \int_{x\in D}(x-\mu_X)^2 f_X(x)\,dx \quad \left[\text{Take } h(x) = (x-\mu_X)^2\right]$$
$$= \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$$

S.D. $:= \sigma_X := +\sqrt{V(X)}$

Verify that, $V(X) = E(X^2) - E^2(X) = E(X^2) - \mu^2$

Defn: $V(h(X)) := \int_{-\infty}^{\infty}\left[h(x) - E(h(X))\right]^2 f(x)\,dx$

Also, $V(aX+b) = a^2\,V(X)$ [Verify!]

eg. 1. $X \sim U(a,b)$

$$E(X^2) = \int^b x^2\cdot\frac{1}{b-a}\cdot dx = \frac{1}{b-a}\cdot\left[\frac{x^3}{3}\right]_a^b$$

$$E(X^2) = \int_a^b x^2 \cdot \frac{1}{b-a} \cdot dx = \frac{1}{b-a} \cdot \left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

$$V(X) = E(X^2) - E^2(X) = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{(b-a)^2}{12} \quad (\text{check !})$$

Lecture 21

Thursday, July 13, 2017 1:00 PM

- We have X. Let $Y = a + bX$. $(b \neq 0)$

If $E_1 = \{a+bX \le y\}$ and $E_2 = \{X \le \frac{y-a}{b}\}$, $E_1 = E_2$ (Same event)

$$F_Y(y) = P(Y \le y) = P(a + bX \le y) = P\left(X \le \frac{y-a}{b}\right) \quad (\text{Here we assume: } b>0)$$

$$= F_X\left(\frac{y-a}{b}\right)$$

Otherwise, $P(a+bX \le y) = P(X \ge \frac{y-a}{b}) = 1 - F_X\left(\frac{y-a}{b}\right)$

Now, $P(X \le \eta_X(p)) = p \Leftrightarrow F_X(\eta_X(p)) = p$

and $P(Y \le \eta_Y(p)) = p \Leftrightarrow F_Y(\eta_Y(p)) = p \Leftrightarrow F_X\left(\frac{\eta_Y(p) - a}{b}\right) = p$

$$\Rightarrow \eta_X(p) = \frac{\eta_Y(p) - a}{b}$$

$$\Rightarrow \eta_Y(p) = a + b\,\eta_X(p)$$

In particular, $\tilde{\mu}_Y = a + b \cdot \tilde{\mu}_X$ (Put $p = 0.5$)

We know: If $X \sim \mathcal{U}(0,1)$, then $Y = a + (b-a) \cdot X \sim \mathcal{U}(a,b)$

$$\Rightarrow \eta_Y(p) = a + (b-a) \cdot \eta_X(p)$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x > 1 \end{cases} \Rightarrow F_X(\eta_X(p)) = p \quad [0 < p < 1]$$

gives us $\eta_X(p) = p$

(Solve for x: $F_X(x) = p$)

$$\boxed{\eta_Y(p) = a + (b-a) \cdot p}$$

In particular, $\tilde{\mu}_Y = a + (b-a)(\cdot 5) = \frac{a+b}{2}$. (as $\tilde{\mu}_X = 0.5$)